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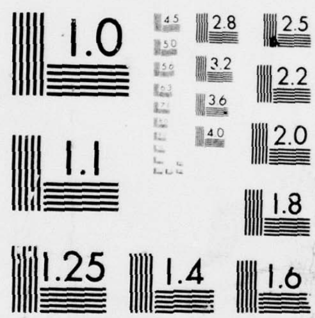

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ON A PROBLEM CONCERNING BAND MATRICES WITH TOEPLITZ INVERSES

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(10) T. N. E. Greville

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ABSTRACT

Greville and Trench have studied the conditions under which a band matrix has a Toeplitz inverse. If $H = (h_{ij})_{i,j=0}^{N}$ has the property that $h_{ij} = 0$ for $j - i > r$ or $i - j > s$, where $r \geq 0$, $s \geq 0$, and $r + s \leq N$, it was shown that H has a Toeplitz inverse if and only if it has a special structure fully characterized by two polynomials, $A(x)$ of degree r and $B(x)$ of degree s. One consequence of the special structure is that H itself is <u>quasi-Toeplitz</u> in the sense that $h_{i+1,j+1} = h_{ij}$ so long as neither of these elements is in the s by r submatrix in the upper left corner or the r by s submatrix in the lower right corner. Suppose H is real and its "Toeplitz part" is given, as well as the fact that it has a Toeplitz inverse, but the two corner submatrices are undetermined. What are the possible choices of the corner submatrices, and what are the corresponding Toeplitz inverses? It is shown in this paper that the number of possible choices is finite, and depends on the pattern of zeros of $A(x)$ and $B(x)$. A criterion is established for defining a "preferred" choice, which is unique if it exists.

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SIGNIFICANCE AND EXPLANATION

In TSR #1879 W. F. Trench and I studied the conditions for a band matrix to have an inverse that is a Toeplitz matrix. We showed that if the band width is not too large, the band matrix must itself have the Toeplitz property (of equality of elements along diagonal lines) except for two submatrices in the upper left and lower right corners. Sometimes one knows the "Toeplitz part" of the band matrix (and the fact that it has a Toeplitz inverse), but the elements of the special corner submatrices are undetermined. Such a problem arose in extending moving-weighted-average smoothing to the extremities of the data (see MRC TSR #1786). In this situation what are the possible choices of the corner submatrices, and what are the corresponding inverses of the entire matrix? This is the problem studied here for real matrices.

It is shown that the number of possible solutions is finite, and a way of finding them is described. A "preferred solution" is defined, which is unique if it exists. An algorithm is given for determining whether there is a preferred solution and for finding it if it exists.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

ON A PROBLEM CONCERNING BAND MATRICES WITH TOEPLITZ INVERSES

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1. Statement of the Problem.

In [2] it was shown that a band matrix having a Toeplitz inverse, when the band width does not exceed the order of the matrix, has a very special structure. More specifically, let $H = (h_{ij})_{i,j=0}^{N}$ be a real matrix, where

$$h_{ij} = 0 \quad \text{for} \quad j - i > r \quad \text{or} \quad i - j > s \,, \tag{1.1}$$

with

$$r \geq 0, \qquad s \geq 0, \qquad r + s \leq N \,. \tag{1.2}$$

Moreover, let

$$H_i(x) = \sum_{j=0}^{N} h_{ij} x^j$$

be the generating function of the elements of the ith row of H. Then it is shown in [2] that H has a Toeplitz inverse if and only if

$$H_i(x) = \begin{cases} x^i A(x) \sum_{\mu=0}^{i} b_\mu x^{-\mu} & (0 \leq i < s) \\ x^i A(x) B(1/x) & (s \leq i \leq N - r) \\ x^i B(1/x) \sum_{\nu=0}^{N-i} a_\nu x^\nu & (N - r < i \leq N) \,, \end{cases} \tag{1.3}$$

where

$$A(x) = \sum_{\nu=0}^{r} a_\nu x^\nu \quad \text{and} \quad B(x) = \sum_{\mu=0}^{s} b_\mu x^\mu$$

are polynomials with real coefficients, $a_o b_o \neq 0$, and $A(x)$ and $x^s B(1/x)$

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have no common zero, real or complex. Equations (1.3) imply that H itself is quasi-Toeplitz in the sense that the Toeplitz property

$$h_{i+1,j+1} = h_{ij} \tag{1.4}$$

holds so long as neither of the elements in (1.4) is in the s by r submatrix in the upper left corner of H or in the r by s submatrix in the lower right corner. In fact, if we define the function $h(x)$ and coefficients h_ν by

$$h(x) = A(x)B(1/x) = \sum_{\nu=-s}^{r} h_\nu x^\nu , \tag{1.5}$$

and if we take $h_\nu = 0$ for $\nu > r$ and $\nu < -s$, then

$$h_{ij} = h_{j-i} , \tag{1.6}$$

so long as h_{ij} is not in one of the two corner submatrices referred to.

This paper deals with the following problem. Suppose H is banded in the sense of (1.1) and (1.2) and is known to have a Toeplitz inverse, and suppose further that the "Toeplitz part" of H is known. In other words, the coefficients h_ν of (1.5) are known. But the elements of the two anomalous corner submatrices are undetermined. What are the possible choices of the elements of these submatrices, and what are the corresponding Toeplitz inverses of H?

This problem has arisen in some research of mine [1] on extending smoothing by moving weighted averages to the extremities of the data. Trench [3] has also encountered band matrices with Toeplitz inverses in a study of prediction of stationary time series.

It will be shown that the choice of the elements of the corner submatrices is in general not unique, but the number of possible choices is finite. In some particular cases there is no solution. It will be shown further that in the more typical case there is in a certain sense a single preferred choice from among the finite number of possible solutions.

2. Existence and Number of Solutions.

In view of (1.5), another way of stating the problem is to say that $h(x) = A(x)B(1/x)$ is specified, but $A(x)$ and $B(1/x)$ are not specified. An equivalent statement is that the polynomial $x^s h(x)$ of degree $r + s$ is

to be expressed as the product of two real polynomials $A(x)$ and $x^s B(1/x)$ of degrees r and s, respectively. If $A(x)$ is multiplied by a real constant and $B(x)$ is divided by the same constant, the function $h(x)$ and the matrix H are not changed, and the same solution is obtained. Thus it is a question of apportioning the $r + s$ zeros of $x^s h(x)$ so that r of them are assigned to $A(x)$ and the remaining s to $x^s B(1/x)$. Clearly this can be done in $\binom{r+s}{r}$ ways, and this is an upper bound to the number of different solutions. However, in general there are some constraints, as we shall now see.

We assume without loss of generality that $a_r b_s \neq 0$. If a_r or b_s vanishes, we have chosen the wrong value of r or s. If a_o or b_o vanishes, or if $A(x)$ and $x^s B(1/x)$ have a common zero, then it is shown in [2] that H is singular, and so it does not have a Toeplitz (or any other) inverse. Thus, these cases are automatically ruled out. If the zeros of $x^s h(x)$ include conjugate complex pairs, they must be assigned together, either to $A(x)$ or to $x^s B(1/x)$, as these are to be real polynomials. If there are multiple zeros, they must be assigned together, either to $A(x)$ or $x^s B(1/x)$; otherwise these polynomials would have a common zero and H would be singular.

Let us now consider some special cases. If r or s is zero, the anomalous corner submatrices are "empty" matrices, and H itself is a Toeplitz matrix. Thus, if $s = 0$, H is an upper-triangular Toeplitz matrix (with nonzero diagonal elements) and we see that its inverse is likewise an upper-Triangular Toeplitz matrix. Similar remarks apply to the case when $r = 0$.

If $r = s = 1$, each of the corner submatrices consists of a single element (the first and last diagonal elements). If the zeros of the quadratic $xh(x)$ are real and unequal, there are two possible choices of the corner elements. If the zeros are real and equal, H as given by (1.3) is singular. If they are complex, there are no real values of the corner elements such that H has a Toeplitz inverse. More generally, there is no real solution if all zeros of $x^s h(x)$ are complex (so that $r + s$ is necessarily even), and r and s are odd.

The case of $r = 2$ and $s = 1$ is more typical. In this case, there is always a solution (except in the case of a single triple zero), since the cubic $xh(x)$ has at least one real zero and can be expressed as the product of a real linear function and a real quadratic. We may take the quadratic factor as $A(x)$ and the linear factor as $xB(1/x)$. If the cubic has three distinct real zeros, there are three possible choices of the corner elements.

Of particular interest is the "symmetrical" case in which

$$(2.1) \qquad r = s \text{ and } h_{-\nu} = h_{\nu} \qquad (\nu = 1,2,\ldots,r) .$$

In this case $h(x)$ has the "reciprocal" property that if λ is a zero, λ^{-1} is a zero. It follows from (2.1) and (1.6) that the "Toeplitz part" of H is symmetric, and it seems not unreasonable to demand that the symmetry extend into the special corner submatrices. In order to accomplish this, one must follow the rule that if the zero λ is assigned to $A(x)$, λ^{-1} must be assigned to $x^sB(1/x)$, and vice versa. If this is done, $A(x)$ and $B(x)$ are the same polynomial, except for multiplication by a real constant, and they can be normalized so that $A(x) = B(x)$. Under these conditions, there is always a solution if $h(x)$ has no zeros on the unit circle. When this is the case, if $2m$ is the number of distinct linear and quadratic factors (irreducible in the real field) of $x^sh(x)$, then the number of possible fully symmetric solutions is 2^m.

If $x^sh(x)$ has zeros on the unit circle, there is never a fully symmetric real solution. If λ is complex and on the unit circle, $\lambda^{-1} = \bar{\lambda}$, and if it is a simple zero of $x^sh(x)$, the aforementioned rule for assignment of reciprocal zeros cannot be followed in such a way that $A(x)$ and $B(x)$ are real polynomials. If λ is a multiple zero (as it must be if it is real), the assignment rule results in $A(x)$ and $x^sB(1/x)$ having a common zero, and H is singular. If symmetry of the special corner submatrices is not demanded (even though the Toeplitz part of H is symmetric), the assignment rule need not be followed, and there may be solutions, even with zeros of $x^sh(x)$ on the unit circle.

3. The Preferred Solution.

In the preceding section we have focused on the band matrix H and the conditions it must satisfy, as shown in [2], in order to have a Toeplitz inverse. Now let us look more closely at the properties of the Toeplitz

inverse H^{-1}. It is clear from (1.3) that the elements of the two special corner submatrices depend only on $A(x)$ and $B(x)$ and do not depend on N. Similarly, the nonzero elements of the "Toeplitz part" of H are merely the coefficients h_ν of (1.5) and (1.6) and the value of N affects only the number of rows and columns containing these elements. It is shown in [2] that a similar remark applies to the inverse H^{-1}.

Let $A(x)$ and $B(x)$ be fixed polynomials of degrees r and s, respectively, such that $a_o b_o \neq 0$ and $A(x)$ and $x^s B(1/x)$ have no common factor, and consider the family of matrices H given by (1.3) for all values of N not less than $r + s$. Consider also the corresponding family of Toeplitz inverses $T = (t_{ij}) = H^{-1}$. Since these are Toeplitz matrices, we have, for any given N,

$$t_{ij} = t_{j-i} \qquad (0 \le i, j \le N) \tag{3.1}$$

for some sequence $\{t_\nu\}_{\nu=-N}^{N}$. We shall use the abbreviated notation,

$$T = T[t_{-N}, t_{-N+1}, \ldots, t_N]$$

to denote the Toeplitz matrix of order $N + 1$ defined by (3.1). It is shown in [2] that the elements of the sequence $\{t_\nu\}$ do not depend on N. Under the stated conditions on $A(x)$ and $B(x)$, the difference equations

$$\begin{cases} \sum_{\nu=0}^{r} a_\nu t_{j-\nu} = 0 & (j > 0) \\ \sum_{\nu=0}^{r} a_\nu t_{-\nu} = b_o^{-1} & \\ \sum_{\mu=0}^{s} b_\mu t_{j+\mu} = 0 & (j < 0), \end{cases} \tag{3.2}$$

uniquely determine a doubly infinite sequence $\{t_\nu\}_{\nu=-\infty}^{\infty}$, from which the elements of $T = H^{-1}$ are drawn, whatever the value of N. Increasing N merely extends the sequence, without changing the values already determined.

It will be noted that, for a given $h(x)$, the effect on H of different choices of $A(x)$ and $B(x)$ involves only the special corner submatrices, but the effect on H^{-1} is to change its elements completely.

Now, consider the series

$$(3.3) \qquad t(x) = \sum_{\nu=-\infty}^{\infty} t_\nu x^\nu ,$$

which we may regard as the generating function of the elements of $T = H^{-1}$. For a fixed $h(x)$, this will be a different series for different choices of $A(x)$ and $B(x)$. It follows from (3.2) that each of these series is a formal "reciprocal" of $h(x)$ in the sense that formal multiplication by $h(x)$ gives unity. Is there some choice of $A(x)$ and $B(x)$ for which (3.3) converges to $[h(x)]^{-1}$ in some region of the complex plane? We shall see that there is at most one such choice.

We shall first show that $t(x)$ can be expressed in the form

$$(3.4) \qquad t(x) = \sigma(x)[B(1/x)]^{-1} + x^s\rho(x)[A(x)]^{-1} ,$$

where $[A(x)]^{-1}$ denotes a formal expansion in nonnegative powers of x, $[B(1/x)]^{-1}$ denotes a formal expansion in nonpositive powers of x, $\rho(x)$ is defined by

$$\rho(x) = x^{-s}A(x) \sum_{\nu=s}^{\infty} t_\nu x^\nu ,$$

and is a polynomial of degree less than r, and $\sigma(x)$ is given by

$$\sigma(x) = B(1/x) \sum_{\nu=-\infty}^{s-1} t_\nu x^\nu ,$$

and is a polynomial of degree less than s. To show this we write

$$(3.5) \qquad t(x) = t_1(x) + t_2(x) ,$$

where

$$t_1(x) = \sum_{\nu=-\infty}^{s-1} t_\nu x^\nu, \qquad t_2(x) = \sum_{\nu=s}^{\infty} t_\nu x^\nu ,$$

and we note that in the product $A(x)t_2(x)$ powers of x with exponents $r + s$ and above vanish because of (3.2). Thus, this product may be written as $x^s\rho(x)$, where $\rho(x)$ is a polynomial of degree less than r, and, in the sense of formal multiplication,

(3.6) $$t_2(x) = t_2(x)A(x)[A(x)]^{-1} = x^s\rho(x)[A(x)]^{-1} .$$

Similarly, all negative powers of x vanish in the product $B(1/x)t_1(x)$, which is therefore a polynomial $\sigma(x)$ of degree less than s, and, consequently,

(3.7) $$t_1(x) = t_1(x)B(1/x)[B(1/x)]^{-1} = \sigma(x)[B(1/x)]^{-1} .$$

Substitution of (3.6) and (3.7) in (3.5) gives (3.4).

Now, the series $[A(x)]^{-1}$ converges in the open disk with centre at the origin and radius equal to the minimum absolute value of the zeros of $A(x)$, while $[B(1/x)]^{-1}$ converges for values of $1/x$ in a disk similarly related to $B(x)$, or, in other words, for values of x outside of a circle with centre at the origin and radius equal to the maximum absolute value of the zeros of $x^sB(1/x)$. Thus (3.4) shows that the series (3.3) converges, if at all, in an open annulus that is the intersection of the two regions mentioned. This intersection is nonempty if and only if all the zeros of $x^sB(1/x)$ are smaller in absolute value than every zero of $A(x)$.

In the light of this discussion we decide to define the preferred solution to the original problem as one based on a choice of $A(x)$ and $B(x)$ such that the series (3.3) converges in some region of the complex plane. It is now clear that there is at most one such solution, and an algorithm is immediately suggested for finding it if it exists.

Let the zeros of $x^sh(x)$ be $\lambda_1,\lambda_2,\ldots,\lambda_{r+s}$, arranged in order of nondecreasing absolute magnitude. A multiple zero is included in the sequence a number of times equal to its multiplicity, and, if there are distinct zeros of equal absolute value, the arrangement of these among themselves is immaterial. Then, if $|\lambda_s| = |\lambda_{s+1}|$, there is no preferred solution to the original problem. If $|\lambda_s| < |\lambda_{s+1}|$, there is a preferred solution, and it is obtained by choosing $B(x)$ so that the zeros of $x^sB(1/x)$ are $\lambda_1,\lambda_2,\ldots,\lambda_s$, and choosing $A(x)$ so that its zeros are $\lambda_{s+1},\lambda_{s+2},\ldots,\lambda_{r+s}$. Then (3.3) converges in the open annulus with centre at the origin bounded by circles with radii $|\lambda_s|$ and $|\lambda_{s+1}|$.

4. Numerical Examples.

In both examples $r = 2$, $s = 1$, and $N = 5$.

Example 1. Let

$$H = \begin{bmatrix} y & z & 3 & 0 & 0 & 0 \\ 8 & 30 & 19 & 3 & 0 & 0 \\ 0 & 8 & 30 & 19 & 3 & 0 \\ 0 & 0 & 8 & 30 & 19 & 3 \\ 0 & 0 & 0 & 8 & 30 & z \\ 0 & 0 & 0 & 0 & 8 & y \end{bmatrix}.$$

Here $xh(x) = 3x^3 + 19x^2 + 30x + 8 = (3x + 1)(x + 2)(x + 4)$. There are three possible choices. We may take $A_1(x) = (x + 2)(x + 4)$, $xB_1(1/x) = 3x + 1$, or $A_2(x) = (3x + 1)(x + 4)$, $xB_2(1/x) = x + 2$, or $A_3(x) = (3x + 1)(x + 2)$, $xB_3(1/x) = x + 4$. The corresponding values of y and z are $y_1 = 24$, $z_1 = 18$; $y_2 = 4$, $z_2 = 13$; $y_3 = 2$, $z_3 = 7$; and the corresponding Toeplitz inverses are

$$T_1 = T\left[-\frac{1}{4455}, \frac{1}{1485}, -\frac{1}{495}, \frac{1}{165}, -\frac{1}{55}, \frac{3}{55}, -\frac{17}{440}, \frac{39}{1760}, -\frac{83}{7040}, \frac{171}{28160}, -\frac{347}{112640}\right],$$

$$T_2 = T\left[\frac{16}{5}, -\frac{8}{5}, \frac{4}{5}, -\frac{2}{5}, \frac{1}{5}, -\frac{1}{10}, \frac{7}{40}, -\frac{79}{160}, \frac{943}{640}, -\frac{11311}{2560}, \frac{135727}{10240}\right],$$

$$T_3 = T\left[-\frac{512}{11}, \frac{128}{11}, -\frac{32}{11}, \frac{8}{11}, -\frac{2}{11}, \frac{1}{22}, \frac{5}{44}, -\frac{41}{88}, \frac{257}{176}, -\frac{1553}{352}, \frac{9329}{704}\right].$$

T_1 is the preferred inverse.

Example 2. Let

$$H = \begin{bmatrix} y & z & 3 & 0 & 0 & 0 \\ 6 & 3 & 6 & 3 & 0 & 0 \\ 0 & 6 & 3 & 6 & 3 & 0 \\ 0 & 0 & 6 & 3 & 6 & 3 \\ 0 & 0 & 0 & 6 & 3 & z \\ 0 & 0 & 0 & 0 & 6 & y \end{bmatrix}.$$

Here $xh(x) = 3x^3 + 6x^2 + 3x + 6 = 3(x + 2)(x^2 + 1)$, and the only possible choice of $A(x)$ and $xB(1/x)$ (except for constant multipliers) is $A(x) = x^2 + 1$ and $xB(1/x) = 3(x + 2)$. This gives $y = 3$ and $z = 0$. The

resulting Toeplitz inverse is $T\left[-\frac{32}{15}, \frac{16}{15}, -\frac{8}{15}, \frac{4}{15}, -\frac{2}{15}, \frac{1}{15}, \frac{2}{15}, -\frac{1}{15}, -\frac{2}{15}, \frac{1}{15}, \frac{2}{15}\right]$. There is no preferred solution, since the only zero of $xB(1/x)$ is $x = -2$, which is greater in absolute value than both zeros of $A(x)$.

Acknowledgment.

I wish to express my deep gratitude to W. F. Trench, who collaborated in the earlier work that underlies the problem treated here, suggested that the number of possible solutions is finite, and encouraged me to pursue this investigation.

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[3] W. F. Trench, Weighting coefficients for the prediction of stationary time series from the finite past, SIAM J. Appl. Math., 15 (1967), 1502-1510.

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20. ABSTRACT *(Continue on reverse side if necessary and identify by block number)*

Greville and Trench have studied the conditions under which a band matrix has a Toeplitz inverse. If $H = (h_{ij})_{i,j=0}^{N}$ has the property that $h_{ij} = 0$ for $j - i > r$ or $i - j > s$, where $r \geq 0$, $s \geq 0$, and $r + s \leq N$, it was shown that H has a Toeplitz inverse if and only if it has a special structure fully characterized by two polynomials, $A(x)$ of degree r and $B(x)$ of

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20. ABSTRACT - Continued

degree s. One consequence of the special structure is that H itself is quasi-Toeplitz in the sense that $h_{i+1,j+1} = h_{ij}$ so long as neither of these elements is in the s by r submatrix in the upper left corner or the r by s submatrix in the lower right corner. Suppose H is real and its "Toeplitz part" is given, as well as the fact that it has a Toeplitz inverse, but the two corner submatrices are undetermined. What are the possible choices of the corner submatrices, and what are the corresponding Toeplitz inverses? It is shown in this paper that the number of possible choices is finite, and depends on the pattern of zeros of $A(x)$ and $B(x)$. A criterion is established for defining a "preferred" choice, which is unique if it exists.